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Optical linewidth of a low density Fermi-Dirac gas

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We study propagation of light in a Fermi-Dirac gas at zero temperature. We analytically obtain the leading density correction to the optical linewidth. This correction is a direct consequence of the quantum statistical correlations of atomic positions that modify the optical interactions between the atoms at small interatomic separations. The gas exhibits a dramatic line narrowing already at very low densities.

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The observation of Bose-Einstein condensation in dilute atomic vapors [1] has spurred much interest in ultracold atomic gases. Another evident milestone of atomic physics would be the cooling of a Fermi-Dirac (FD) gas to the quantum degenerate regime. So far all probing of atomic Bose-Einstein condensates has been done optically, and obviously optical detection could also play an important role in the experimental studies of FD gases. Appropriately, theoretical studies of the FD gases are experiencing a renaissance [2–6].

In this paper we investigate propagation of low-intensity light in a FD gas in the limit of low atom density. We derive the leading quantum statistical correction to the standard column-density refractive index analytically, by legitimately ignoring collective linewidths and line shifts generated in the processes in which a photon is repeatedly scattered between the same atoms. A fermion gas exhibits a striking line narrowing characteristic of the FD statistics, which behaves as $\rho^{2/3}$ at low atom densities.

In the dipole approximation it is advantageous to transform the Hamiltonian into the *length* gauge by the Power-Zienau-Woolley transformation [7]. Then the positive frequency component of the electric field $\mathbf{E}^+$ may be expressed [8,9] in terms of the positive frequency components of the driving electric displacement, $\mathbf{D}_F^+$, and of the source field radiated by atomic polarization, $\mathbf{P}^+$, as

$$\epsilon_0 \mathbf{E}^+(\mathbf{r}) = \mathbf{D}_F^+(\mathbf{r}) + \frac{1}{i\kappa}\int d^3r'\, \mathsf{G}(\mathbf{r}-\mathbf{r}')\,\mathbf{P}^+(\mathbf{r}')\,, \tag{1a}$$

$$\mathsf{G}_{ij}(\mathbf{r}) = i\kappa\left\{\left[\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} - \delta_{ij}\boldsymbol{\nabla}^2\right]\frac{e^{ikr}}{4\pi r} - \delta_{ij}\delta(\mathbf{r})\right\}. \tag{1b}$$

Here $k = \Omega/c$, Ω is the frequency of the driving field, and the scalar constant $\kappa = \mathcal{D}^2/\hbar\epsilon_0$ is defined in terms of the reduced dipole moment matrix element $\mathcal{D}$. The monochromatic dipole radiation kernel $\mathsf{G}(\mathbf{r})$ coincides with the corresponding classical expression [10]. In second quantization the polarization reads

$$\mathbf{P}^+(\mathbf{r}) = \mathbf{d}_{ge}\psi_g^\dagger(\mathbf{r})\psi_e(\mathbf{r})\,. \tag{2}$$

Here ψ_g and ψ_e are the ground state and the excited state atom field operators in the Heisenberg picture, and $\mathbf{d}_{ge}$ is the dipole matrix element for the transition $g \to e$. For simplicity, we consider here two-level atoms with just a single ground state $|g\rangle$ and one excited state $|e\rangle$, using a constant real vector $\mathbf{d}$ (such that $\mathcal{D} = |\mathbf{d}|$) as the dipole matrix element.

While Eqs. (1) describe the scattered light in a medium, in general, with a small atom-light detuning and for a dense atomic sample, there is no easy way to find the polarization $\mathbf{P}^+(\mathbf{r})$. By making a field theory version of the Born and Markov approximations, we have derived a hierarchy of equations of motion for correlation functions that contain one excited-atom field and one, three, five, etc., ground state atom fields, for the limit of low light intensity [8]. In the present case of two-level atoms the hierarchy reads

$$\begin{aligned}
&\dot{\mathbf{P}}_l(\mathbf{r}_1,\cdots,\mathbf{r}_{l-1};\mathbf{r}_l) = (i\delta-\gamma)\mathbf{P}_l(\mathbf{r}_1,\cdots,\mathbf{r}_{l-1};\mathbf{r}_l)\\
&+\sum_{k=1}^{l-1}\mathsf{P}\cdot\mathsf{G}'(\mathbf{r}_l-\mathbf{r}_k)\mathbf{P}_l(\mathbf{r}_1,\cdots,\mathbf{r}_{k-1},\mathbf{r}_{k+1},\cdots,\mathbf{r}_l;\mathbf{r}_k)\\
&+i\kappa\rho_l(\mathbf{r}_1,\cdots,\mathbf{r}_l)\mathsf{P}\cdot\mathbf{D}_F^+(\mathbf{r}_l)\\
&+\int d^3r_{l+1}\,\mathsf{P}\cdot\mathsf{G}'(\mathbf{r}_l-\mathbf{r}_{l+1})\mathbf{P}_{l+1}(\mathbf{r}_1,\cdots,\mathbf{r}_l;\mathbf{r}_{l+1})\,,
\end{aligned} \tag{3}$$

where $\gamma = \mathcal{D}^2k^3/(6\pi\hbar\epsilon_0)$ denotes the spontaneous linewidth, and δ is the atom-light detuning. We have defined a projection operator

$$\mathsf{P} \equiv \frac{\mathbf{dd}}{|\mathbf{d}|^2}\,, \tag{4}$$

whose purpose is to eliminate all but the two atomic states from consideration, and the correlation functions

$$\begin{aligned}
&\mathbf{P}_l(\mathbf{r}_1,\cdots,\mathbf{r}_{l-1};\mathbf{r}_l)\\
&\equiv\langle\psi_g^\dagger(\mathbf{r}_1)\cdots\psi_g^\dagger(\mathbf{r}_{l-1})\mathbf{P}^+(\mathbf{r}_l)\psi_g(\mathbf{r}_{l-1})\cdots\psi_g(\mathbf{r}_1)\rangle, \qquad (5a)\\
&\rho_l(\mathbf{r}_1,\cdots,\mathbf{r}_l)\equiv\langle\psi_g^\dagger(\mathbf{r}_1)\cdots\psi_g^\dagger(\mathbf{r}_l)\psi_g(\mathbf{r}_l)\cdots\psi_g(\mathbf{r}_1)\rangle. \qquad (5b)
\end{aligned}$$

The quantity $\mathbf{P}_l$ reflects correlations between the dipole moment of one atom and the positions of $l-1$ other atoms, and ρ_l is simply the density correlation function for l ground state atoms.

The terms in the sum on the right-hand side of Eqs. (3) represent processes in which the l atoms at $\mathbf{r}_1,\ldots,\mathbf{r}_l$ repeatedly exchange photons. Such processes are the microscopic mechanism for collective linewidths and line

shifts. The integral stands for a process in which yet another atom shines its light on the atom at $\mathbf{r}_l$.

Due to the resulting divergent dipole-dipole interactions, all correlation functions $\mathbf{P}_l$ vanish whenever two position arguments are the same [9]. The Lorentz-Lorenz local-field correction follows mathematically from this observation. Moreover, without changing the outcome of the hierarchy, we may, and will, remove all contact interactions between different atoms in Eqs. (3) by introducing the field propagator G' defined by

$$\mathsf{G}'_{ij}(\mathbf{r}) = \mathsf{G}_{ij}(\mathbf{r}) + i\kappa\delta_{ij}\delta(\mathbf{r})/3\,. \tag{6}$$

This definition indicates that the integral of G' over an infinitesimal volume enclosing the origin vanishes.

The coupled theory for light and matter fields [Eqs. (1) and (3)] may be solved, in principle exactly, by means of stochastic simulations [6]. This is because the correlation hierarchy (3) is the same as the hierarchy describing the classical electrodynamics of charged harmonic oscillators with the position correlations ρ_l. By synthesizing a stochastic ensemble of samples of dipoles that have the position correlation functions ρ_l and calculating the ensemble-averaged response to classical light, we have a solution to Eqs. (3). Unfortunately, such simulations are demanding on computer time. The computations of Ref. [6] were therefore performed within a one-dimensional (1D) model electrodynamics. While the predictive power of 1D electrodynamics may be questioned, the simulation results for a FD gas at $T = 0$ show clear signatures of the quantum statistics: Even in the limit of zero density, the optical linewidth of the FD gas is only half of the resonance linewidth of an isolated atom [6].

The 1D simulations have also allowed us to test predictions of the density expansion introduced by Morice *et al.* [11] in their studies of the optical response of a quantum degenerate Bose-Einstein gas. At least in one dimension this expansion is in semi-quantitative agreement with numerical simulations, and in the low-density limit the agreement is excellent [6]. With this in mind, we venture to use the approximation of Morice *et al.* [11] to truncate the correlation hierarchy (3) also in the present three-dimensional case.

We consider the steady-state solution of (3). The atoms are assumed to fill the half-infinite space $z > 0$ with a constant density ρ. The incoming free field is written $\mathbf{D}_F(\mathbf{r}) = D_F\,\hat{\mathbf{e}}\exp(ikz)$, and we assume that $\hat{\mathbf{e}}||\mathbf{d}$. The hierarchy of equations (3) is truncated by writing [11]

$$\mathbf{P}_3(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3) \simeq \frac{\rho_2(\mathbf{r}_1,\mathbf{r}_2)}{\rho}\mathbf{P}_2(\mathbf{r}_2;\mathbf{r}_3)\,. \tag{7}$$

The pair correlation function is written in the form

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2) = \rho^2\left[1+\varphi(\mathbf{r}_1-\mathbf{r}_2)\right]. \tag{8}$$

By introducing the dimensionless quantities

$$\bar{\delta} \equiv \delta/\gamma, \quad \bar{\alpha} \equiv -\frac{6\pi}{\bar{\delta}+i}, \quad \bar{\rho} \equiv \rho/k^3, \quad \bar{\mathsf{G}}'(\mathbf{r}) \equiv \frac{\mathsf{P}\cdot\mathsf{G}'(\mathbf{r})}{i\kappa k^3}\,, \tag{9}$$

where $\bar{\alpha}$ denotes the dimensionless atomic polarizability, and the ansatz $\mathbf{P}_1(\mathbf{r}) = P\,\hat{\mathbf{e}}\exp(ik'z)$ with $\mathrm{Im}(k') > 0$, we obtain the susceptibility of the sample as

$$\chi = \frac{k'^2}{k^2} - 1 = \frac{\bar{\alpha}\bar{\rho}}{1-\bar{\alpha}\bar{\rho}/3 + C}\,, \tag{10}$$

with

$$C = -\bar{\rho}\int d^3\bar{r}\,\hat{\mathbf{e}}^*\cdot\left[\frac{\bar{\alpha}^3\bar{\mathsf{G}}'^3 e^{-i\bar{z}} + \bar{\alpha}^2\bar{\mathsf{G}}'^2}{1-\bar{\alpha}^2\bar{\mathsf{G}}'^2}\right]\cdot\hat{\mathbf{e}}$$
$$-\bar{\rho}\int d^3\bar{r}\,\varphi\hat{\mathbf{e}}^*\cdot\left[\frac{\bar{\alpha}\bar{\mathsf{G}}' e^{-i\bar{z}} + \bar{\alpha}^2\bar{\mathsf{G}}'^2}{1-\bar{\alpha}^2\bar{\mathsf{G}}'^2}\right]\cdot\hat{\mathbf{e}}\,. \tag{11}$$

Here we use the dimensionless integration variable $\bar{\mathbf{r}} = k\mathbf{r}$. The quantity C, and hence also k', have been forced to be independent of position by essentially ignoring the effects of the surface of the atomic sample [6].

The second term in the denominator in Eq. (10) gives the Lorentz-Lorenz shift. In the absence of the C term the electric susceptibility is the standard column density result augmented with a local-field correction. Atom statistics and atom-field collective effects are encapsulated in the integral C. The expansion of Ref. [11] is such that the parameter C takes into account quantum statistical position correlations between any pair of atoms and the exchange of photons between them to arbitrary order, but ignores all repeated photon exchange involving more than two atoms. Correspondingly, one may expand the functions inside the integrals in Eq. (11) as power series in $\bar{\alpha}\bar{\mathsf{G}}'$, and interpret the nth order as a contribution in which a photon is radiated between a pair of atoms n times.

In this paper we only consider the leading order in the modifications of the optical response of an atom due to the presence of the other atoms, and write

$$C \simeq -\bar{\rho}\bar{\alpha}\int d^3\bar{r}\,e^{-i\bar{z}}\varphi(\bar{\mathbf{r}})\,\hat{\mathbf{e}}^*\cdot\bar{\mathsf{G}}'(\bar{\mathbf{r}})\cdot\hat{\mathbf{e}}\,. \tag{12}$$

The expression (12) arises from processes in which any "probe" dipole is subject to the external driving field, and in addition to the primary radiation from the other dipoles. Collective linewidths and line shifts, processes that involve the repeated scattering of a photon between the same atoms, are ignored. For uncorrelated locations of the dipoles with $\varphi = 0$, the effects of the primary radiation from the other dipoles on a probe dipole average to zero. However, for the FD statistics Eq. (12) gives a nontrivial result. This reflects the short-range ordering, within the correlation length, of the atoms in the gas.

Fermions at $T = 0$ fill the Fermi sphere $\bar{n}_{\mathbf{k}} = \Theta(k_F - |\mathbf{k}|)$, with the Fermi wave number $k_F = (6\pi^2\rho)^{1/3}$. In this case we may evaluate the pair correlation function

[Eq. (8)] in closed form. In the thermodynamic limit the result is

$$\varphi(\mathbf{r}) = -\frac{9}{k_F^4 r^4}\left[\frac{\sin k_F r}{k_F r} - \cos k_F r\right]^2 . \qquad (13)$$

We also set the polarization of the incoming light field to be in the x direction. The propagator in Eq. (12) then reads

$$\hat{\mathbf{e}}_x^* \cdot \bar{\mathsf{G}}'(\bar{\mathbf{r}}) \cdot \hat{\mathbf{e}}_x = \frac{e^{i\bar{r}}}{4\pi}\left[(1-\sin^2\theta\cos^2\phi)\frac{1}{\bar{r}} + (3\sin^2\theta\cos^2\phi - 1)(\frac{1}{\bar{r}^3} - \frac{i}{\bar{r}^2})\right] . \qquad (14)$$

After inserting Eqs. (13) and (14) into Eq. (12) we obtain a (complicated) analytical expression, whose density expansion reads

$$C = \frac{3i}{10}\left(\frac{\pi}{6}\right)^{1/3}\bar{\alpha}\bar{\rho}^{2/3} + \mathcal{O}(\bar{\rho}) . \qquad (15)$$

In our 1D electrodynamics, the entire expression (11) may be integrated analytically for a FD gas at $T = 0$ [6]. It is then easy to see that the lowest-order density contribution is correctly introduced by the expansion (12). This is also the case in three dimensions, although the demonstration is more indirect. First, the terms in Eq. (11) that do not depend on $\varphi(r)$ are linearly proportional to $\bar{\rho}$. Second, expanding the contributions to Eq. (11) that *do* depend on φ into a series of $\bar{\alpha}\bar{\mathsf{G}}'$, for orders beyond the one included in (12) we find radial integrals of the form

$$\int d\bar{r}\,\bar{r}^2 \frac{e^{\xi i\bar{r}}}{\bar{r}^{n-1}}\varphi(\bar{r}) \propto \mathcal{O}(\bar{\rho}),$$
$$\int d\bar{r}\,\bar{r}^2 \frac{1}{\bar{r}^n}\varphi(\bar{r}) \propto \mathcal{O}(\bar{\rho})^{n/3} .$$

Here ξ are integers independent of $\bar{\rho}$, and $n \geq 3$. Most of these integrals formally diverge at the origin, but in a manner that must eventually cancel to give a finite result (11). Besides, the divergences do not depend on density. Omitting the divergences, the integrals scale with density as indicated. All told, Eq. (12) not only represents the lowest-order correction to the optical properties in terms of the number of microscopic optical interaction processes between the atoms, but it also correctly gives the leading density correction $\propto \bar{\rho}^{2/3}$ to the optical response.

We have plotted the linewidth and the line shift, including both the effect of the FD statistics and the Lorentz-Lorenz shift,

$$\Gamma = \gamma\left[1 - 6\pi\mathrm{Im}\left(\frac{C}{\bar{\alpha}}\right)\right],$$
$$\Delta = \gamma\left[2\pi\bar{\rho} - 6\pi\mathrm{Re}\left(\frac{C}{\bar{\alpha}}\right)\right] , \qquad (16)$$

as a function of density in Fig. 1 using the full form of C from of Eq. (12). A zero-temperature FD gas exhibits a striking linewidth narrowing already at low densities. For $\bar{\rho} = 1.5\times10^{-3}$ the optical linewidth of the gas is $\Gamma \simeq 0.94\,\gamma$, and with $\bar{\rho} = 1.5\times10^{-2}$ we have $\Gamma \simeq 0.79\,\gamma$. For the 767 nm transition in ^{40}K [5] the corresponding densities would be $\rho \simeq 8.2\times10^{11}$ cm^{-3} and $\rho \simeq 8.2\times10^{12}$ cm^{-3}. At $\bar{\rho} = 0.1$ ($\rho \simeq 5.5\times10^{13}$ cm^{-3}) the optical linewidth of a FD gas would be approximately half of the linewidth of an isolated atom. However, at the latter density we may already have to consider collective linewidths and line shifts to obtain a reliable quantitative prediction, a task we do not undertake in the present paper. From Fig. 1(b), the line shift is negative at low densities and completely vanishes at $\bar{\rho} \simeq 0.03$. At higher densities the line shift turns positive.

The dramatic line narrowing may be attributed to the regular spacing between the atoms characteristic of the FD statistics. As discussed in Ref. [6], the mechanism is particularly transparent in one dimension. An alternative description of the line narrowing may be obtained in the momentum representation. At $T = 0$ the fermions fill the Fermi sphere. Due to the Pauli exclusion principle only either strict forward scattering or scattering events that take the recoiling atom out of the Fermi sea are allowed. The change of the wave vector of an atom upon scattering satisfies $|\Delta\boldsymbol{\kappa}| = 2k\sin(\theta/2)$, where θ is the scattering angle for photons. All atoms are scattered out of the Fermi sea if $|\Delta\boldsymbol{\kappa}| > 2k_F$. Thus, we see that for the photon scattering angles θ satisfying

$$\sin(\theta/2) > k_F/k = (6\pi^2\bar{\rho})^{1/3} , \qquad (17)$$

scattering is not inhibited by the FD statistics. On the other hand, for $\sin(\theta/2) < k_F/k$ some recoil events would lead to an already occupied state in the Fermi sea, and are forbidden. The suppression of light scattering is strongest in the near-forward direction corresponding to small values of θ. When the density is increased, at $\bar{\rho} \geq 1/(6\pi^2)$ we have $k_F \geq k$, and scattering is at least partially suppressed in all nonforward directions. Correspondingly, Fig. 1 shows that $\bar{\rho} = 1/(6\pi^2) \simeq 0.017$ is a relevant scale for the density.

It is instructive to note the difference between different atom statistics. For the Bose-Einstein condensate the standard factorization of the correlation functions, $\rho_l = \rho^l$, corresponds to an uncorrelated atomic sample, and gives $\varphi = 0$. The leading correction to the standard column density linewidth therefore results from the *cooperative* optical effects, the collective optical linewidths and line shifts, and it is proportional to atom density, just as the Lorentz-Lorenz shift. FD statistics is different because the correlations have a length scale k_F^{-1} that enters the argument, *and* the length scale itself depends on density: $k_F^{-1} \propto \bar{\rho}^{-1/3}$. The result is that, at low densities, the effects of the FD statistics dominate over the Lorentz-Lorenz shift. A Maxwell-Boltzmann ideal gas has another nontrivial correlation function φ, but the

length scale is determined by temperature and does not depend on density. The leading density correction to the optical response is then once more proportional to atom density.

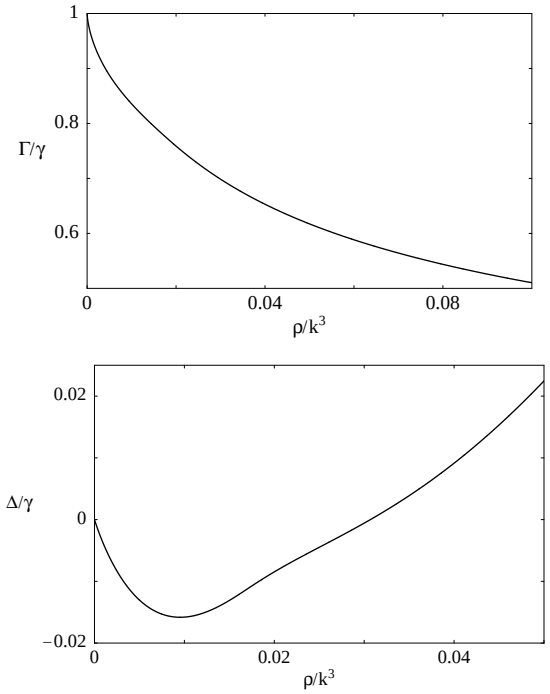

FIG. 1. The optical (a) linewidth and (b) the line shift of a Fermi-Dirac gas as a function of the atomic density per cubic optical wave number of the driving light $\bar{\rho} = \rho/k^3$.

We have assumed a homogeneous FD gas in our analysis. Now, a FD gas may be considered locally homogeneous [3] when the length scale over which the density varies is much larger than the spatial correlation length. Given the length scale of a harmonic trap $l = (\hbar/m\omega)^{1/2}$ and the correlation length $1/k_F = (6\pi^2\rho)^{-1/3}$ from Eq. (13), the criterion reads $l^3\rho \gg 1$. A simple dimensional argument shows that this is the same as requiring that the number of trapped atoms be much larger than one. Furthermore, if the size scale of the atomic sample is much larger than the wavelength of light, $l \gg \lambda$, it is reasonable to expect that the refractive index as appropriate for the local density applies in the bulk of the gas.

In conclusion, we have studied propagation of light in a FD gas. We have discussed quantum statistical corrections to the refractive index, and calculated the leading density correction to the standard column density susceptibility. Already at low densities, fermions exhibit a dramatic narrowing of the resonance line. This might serve as a signature of quantum degeneracy in a cold FD gas.

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